

MAXIMAL n -ORTHOGONAL MODULES FOR SELFINJECTIVE ALGEBRAS

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ABSTRACT. Let A be a selfinjective algebra. We show that, for any $n \geq 1$, maximal n -orthogonal A -modules (in the sense of Iyama) rarely exist. More precisely, we prove that if A admits a maximal n -orthogonal module, then *all* A -modules are of complexity at most 1.

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1. INTRODUCTION

Recently, O. Iyama introduced maximal n -orthogonal modules for finite-dimensional algebras, and developed an extensive theory [17], [18]. One aspect is a 'higher Auslander correspondence', generalizing the famous one-one correspondence between algebras of finite representation type and Auslander algebras, that is, algebras of global dimension at most 2 and dominant dimension at least 2.

The existence of a maximal n -orthogonal module of an algebra A has very striking consequences for the homological properties of A and its modules. In particular it follows that then the representation dimension of A (for background see [1]) is at most $n + 2$. Of special interest is the case $n = 1$. Maximal 1-orthogonal modules are known to exist for certain algebras of finite representation type, and also for preprojective algebras [15]. If A has a maximal 1-orthogonal module then the representation dimension is at most 3. Using a result of K. Igusa and G. Todorov [16], this implies that the famous finitistic dimension conjecture holds for A , that is, there is a finite bound on the projective dimensions of A -modules of finite projective dimension.

If there are maximal 1-orthogonal modules, then usually they are not unique. However, Iyama showed that the endomorphism rings of any two maximal 1-orthogonal modules of a fixed algebra are derived equivalent ([18], 5.3.3). Moreover, he established a striking 'exchange rule': taking an indecomposable summand X' of a maximal 1-orthogonal module, there is at most one indecomposable module T not isomorphic to X' which can be substituted

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for X' giving another maximal 1-orthogonal module (a proof can also be found in [15], 4.5).

Maximal 1-orthogonal modules are crucial for the work on cluster algebras of C. Geiß, B. Leclerc and J. Schröer [15]. Cluster algebras were introduced by Fomin and Zelevinsky in [14] to study canonical bases of quantum groups; a central feature is the introduction of an 'exchange graph'. In the approach of [15], the exchange property of maximal 1-orthogonal modules for preprojective algebras describes the exchange graph for the associated cluster algebra. For details, see [15].

Because of these results, it would be very interesting to know how common maximal 1-orthogonal modules are. However we discovered that for selfinjective algebras they are very rare, and perhaps occur only for the known cases of finite representation type and preprojective algebras. The aim of this note is to give a proof of this, and also show that for any $n \geq 1$ maximal n -orthogonal modules are rare for selfinjective algebras. This will also show that preprojective algebras play a very special role.

We recall the definition of a maximal n -orthogonal module for a finite-dimensional algebra A , due to Iyama [17]. For an A -module X , we denote by $\text{add}(X)$ the full subcategory of the module category $\text{mod } A$ whose objects are direct summands of direct sums of copies of X . A (finitely generated) A -module X is called *maximal n -orthogonal* if for every A -module M the following three conditions are equivalent

- (i) $\text{Ext}_A^i(M, X) = 0$ for all $1 \leq i \leq n$.
- (ii) $\text{Ext}_A^i(X, M) = 0$ for all $1 \leq i \leq n$.
- (iii) $M \in \text{add}(X)$.

For more details and some examples illustrating this concept, we refer to Section 2.2 below.

The following is the main result of this note, showing that only very few selfinjective algebras can possibly admit maximal n -orthogonal modules.

Theorem 1.1. *Let A be a selfinjective algebra, and suppose that for some $n \geq 1$, there exists a maximal n -orthogonal A -module. Then all A -modules have complexity at most 1.*

Recall that complexity of a module measures the growth of its minimal projective resolution. For selfinjective algebras, the most common modules that have complexity ≤ 1 are the Ω -periodic modules (here ΩM is the kernel of a minimal projective cover of the module M , and M is Ω -periodic if $\Omega^k(M) \cong M$ for some $k \geq 1$).

Let us point out that the existence of maximal 1-orthogonal modules for finite-dimensional preprojective algebras is perfectly in line with our above results. In fact, for preprojective algebras of Dynkin type all modules are Ω -periodic, of period at most 6 (an unpublished result of C.M. Ringel and A. Schofield; for a proof see for instance [12], [13], or [7]).

Algebras are in this paper assumed to be finite-dimensional algebras over a field K . All modules are finitely generated right modules, and $\text{mod } A$ denotes the category of finitely generated A -modules.

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2. BACKGROUND AND PRELIMINARIES

2.1. Homological algebra for selfinjective algebras. Let A be a finite-dimensional selfinjective algebra, so that projective modules and injective modules are the same. For a module M , we have ΩM , the kernel of a minimal projective cover, and we also have $\Omega^{-1}M$, the cokernel of an injective hull. Then Ω and Ω^{-1} induce mutually inverse equivalences of the stable module category of A (see for example [3], Chapter IV). Recall that the stable module category $\underline{\text{mod}} A$ has the same objects as $\text{mod } A$, and the morphisms $\underline{\text{Hom}}_A(M, N)$ are equivalence classes of module homomorphisms modulo those factoring through a projective A -module. In particular we have for all $k \geq 1$ that

$$\text{Ext}_A^k(M, N) \cong \underline{\text{Hom}}_A(\Omega^k M, N) \cong \underline{\text{Hom}}_A(M, \Omega^{-k} N).$$

2.2. Maximal n -orthogonal modules. For the convenience of the reader, we restate here Iyama's definition of a maximal n -orthogonal module for a finite-dimensional algebra, as already given in the introduction.

An A -module X is called *maximal n -orthogonal* if for every A -module M the following three conditions are equivalent

- (i) $\text{Ext}_A^i(M, X) = 0$ for all $1 \leq i \leq n$.
- (ii) $\text{Ext}_A^i(X, M) = 0$ for all $1 \leq i \leq n$.
- (iii) $M \in \text{add}(X)$.

If X is maximal n -orthogonal then all projective indecomposable A -modules and all injective indecomposable A -modules must be summands of X . Moreover, X does not have self-extensions, that is, $\text{Ext}_A^i(X, X) = 0$ for $i = 1, \dots, n$.

We are interested in studying such modules when the algebra A is selfinjective but not semisimple. In that case, X must have at least one indecomposable summand which is not projective (and injective). Namely, otherwise every indecomposable A -module M would have to be a summand of X since $\text{Ext}_A^i(X, M) = 0$ for $i = 1, \dots, n$ and then every indecomposable A -module would be projective and then even simple, and A would be semisimple.

Furthermore, if X is a maximal n -orthogonal module of a selfinjective algebra A then so is $\Omega^t X \oplus A$ for any $t \in \mathbb{Z}$.

2.2.1. *Some examples.* Here are some easy explicit examples to illustrate the concept.

(1) First, let $A = K[T]/(T^t)$, a truncated polynomial ring. Then A has t indecomposable modules (up to isomorphism), of dimensions $1, 2, \dots, t$. But all non-projective indecomposable A -modules have self-extensions. Hence, there is no maximal n -orthogonal A -module for any $n \geq 1$.

(2) Let Q be the following quiver

$$1 \bullet \rightleftarrows \bullet 2$$

Set $A = KQ/\text{rad}^2(KQ)$, a four-dimensional selfinjective algebra. This algebra has precisely four indecomposable modules, namely two simple modules S_1 and S_2 , and two indecomposable projectives P_1 and P_2 with P_i the projective cover of S_i . Let $X := P_1 \oplus P_2 \oplus S_1$, then X is a maximal 1-orthogonal A -module. In fact, $\text{Ext}_A^1(X, S_2) \cong \text{Ext}_A^1(S_1, S_2) \neq 0$ and $\text{Ext}_A^1(S_2, X) \cong \text{Ext}_A^1(S_2, S_1) \neq 0$.

Then $X' := P_1 \oplus P_2 \oplus S_2$ also is a maximal 1-orthogonal A -module, since $X' = \Omega X \oplus A$.

(3) We consider the quiver Q as in (2), and now let $A = KQ/\text{rad}^3(KQ)$. Then the selfinjective algebra A has six indecomposable modules, namely the simple modules S_1, S_2 , their projective covers P_1, P_2 , and furthermore a 2-dimensional module $U_{1,2}$ with top S_1 , and a 2-dimensional module $U_{2,1}$ with top S_2 . Note that $U_{1,2} \cong \Omega S_2$ and $U_{2,1} \cong \Omega S_1$. Suppose we have a maximal 1-orthogonal A -module X , then X must have at least one non-projective indecomposable summand. We may assume that it has a simple summand (otherwise we replace X by $\Omega^{-1}X \oplus A$). At most one of the simples can be a summand of X (since S_1 and S_2 have a non-split extension). Suppose, say, S_1 is a summand of X . Now, for any indecomposable non-projective module $M \neq S_1$, we have $\text{Ext}_A^1(M, S_1) \neq 0$ or $\text{Ext}_A^1(S_1, M) \neq 0$. So X can have no further non-projective summands. On the other hand,

$$\text{Ext}_A^1(X, U_{1,2}) \cong \text{Ext}_A^1(S_1, U_{1,2}) \cong \underline{\text{Hom}}_A(U_{2,1}, U_{1,2}) = 0$$

a contradiction, since X is assumed to be maximal 1-orthogonal.

(4) For any natural number n there exists a selfinjective algebra A with an n -orthogonal module, as the following example shows.

For any $n \geq 1$, let Q be the cyclic (oriented) quiver with $m := 2n+2$ vertices. Then consider the selfinjective algebra $A = KQ/\text{rad}^2(KQ)$. Note that the indecomposable A -modules are the projectives P_0, P_1, \dots, P_{m-1} , and the simple modules S_0, S_1, \dots, S_{m-1} . We label the simple modules so that P_i has socle S_{i+1} , with indices taken modulo m . Then it is straightforward to check that the module

$$X := P_0 \oplus P_1 \oplus \dots \oplus P_{m-1} \oplus S_0 \oplus S_{n+1}$$

is a maximal n -orthogonal A -module.

In general, it is not at all easy to decide whether or not maximal n -orthogonal modules exist. In [18], O. Iyama discusses the case $n = 1$ for selfinjective algebras of finite representation type, and gives a combinatorial reformulation in terms of certain triangulations of regular m -gons. For preprojective algebras, the existence of maximal 1-orthogonal modules is proved in [15].

2.3. Auslander-Reiten formula [2]. Let A be any finite-dimensional algebra, and let $\tau = DTr$ be the Auslander-Reiten translation. When A is selfinjective, we have $\tau \cong \Omega^2\nu$ where ν is a Nakayama automorphism of A , see [3] Chapter IV, 3.7. Then for any A -modules M, N we have

$$D\operatorname{Ext}_A^1(M, N) \cong \underline{\operatorname{Hom}}_A(\tau^{-1}N, M).$$

We will use freely that Ω and ν commute for a selfinjective algebra A .

3. PERIODICITY OF n -ORTHOGONAL MODULES

The following theorem is the first crucial step in proving our main result.

Theorem 3.1. *Let A be a selfinjective algebra, and, for some $n \geq 1$, let X be a maximal n -orthogonal A -module. If Y is a direct summand of X then so is $\Omega^{n+2}\nu Y$. Hence every non-projective indecomposable summand of X is $\Omega^{n+2}\nu$ -periodic.*

Before embarking on the proof, we make an easy but useful observation.

Lemma 3.2. *Let A be a selfinjective algebra. For any A -module M and any $i \geq 1$ we have an isomorphism of vector spaces*

$$\operatorname{Ext}_A^i(M, N) \cong \operatorname{Ext}_A^1(N, \Omega^{i+2}\nu M).$$

Proof. Using the Auslander-Reiten formula 2.3 and the formula 2.1 we get

$$\begin{aligned} \operatorname{Ext}_A^i(M, N) &\cong \underline{\operatorname{Hom}}_A(\Omega^i M, N) \cong \underline{\operatorname{Hom}}_A(\Omega M, \Omega^{-i+1}N) \\ &\cong \operatorname{Ext}_A^1(M, \Omega^{-i+1}N) \cong \underline{\operatorname{Hom}}_A(\tau^{-1}\Omega^{-i+1}N, M) \\ &\cong \underline{\operatorname{Hom}}_A(\Omega N, \tau\Omega^i M) \cong \operatorname{Ext}_A^1(N, \tau\Omega^i M) \\ &\cong \operatorname{Ext}_A^1(N, \Omega^{i+2}\nu M) \end{aligned}$$

where for the last isomorphism we use the fact that, for A selfinjective algebra, one has $\tau = \Omega^2\nu$. \square

Now we are in the position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let X be a maximal n -orthogonal module for the selfinjective algebra A . We consider the A -module $\Omega^{n+2}\nu X$. For any i such that $0 < i \leq n$ we obtain

$$\begin{aligned} \operatorname{Ext}_A^i(X, \Omega^{n+2}\nu X) &\cong \underline{\operatorname{Hom}}_A(\Omega^i X, \Omega^{n+2}\nu X) \cong \underline{\operatorname{Hom}}_A(\Omega X, \Omega^{-i+n+3}\nu X) \\ &\cong \operatorname{Ext}_A^1(X, \Omega^{-i+n+3}\nu X) \cong \operatorname{Ext}_A^{-i+n+1}(X, X) \end{aligned}$$

where the last isomorphism comes from Lemma 3.2. Note that the superscripts $-i + n + 1$ run through the set $\{1, 2, \dots, n\}$. Since X is maximal

n -orthogonal, we conclude that $\text{Ext}_A^{-i+n+1}(X, X) = 0$ for all $i = 1, \dots, n$, which means that $\text{Ext}_A^i(X, \Omega^{n+2}\nu X) = 0$ for $i = 1, \dots, n$. Using again that X is maximal n -orthogonal we deduce that $\Omega^{n+2}\nu X \in \text{add}(X)$.

In particular, if Y is an indecomposable direct summand of X , then also $\Omega^{n+2}\nu Y$ is a direct summand of X .

This means that $\Omega^{n+2}\nu$ permutes the indecomposable non-projective summands of X (recall that $\Omega^{n+2}\nu$ induces a permutation on the set of non-projective indecomposable A -modules). But X has by definition only finitely many indecomposable summands. Hence some power of $\Omega^{n+2}\nu$ is the identity permutation on the non-projective summands of X , that is, X is $\Omega^{n+2}\nu$ -periodic. \square

4. COMPLEXITY AT MOST 1

4.1. Complexity. Let A be a finite-dimensional algebra. For any A -module M , let

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

be a minimal projective resolution. The complexity of M measures the rate of growth of the terms of such a resolution. More precisely, the *complexity of M* is defined as

$$\text{cx}(M) := \inf\{b \in \mathbb{N}_0 \mid \exists c > 0 : \dim P_n \leq c n^{b-1} \text{ for all } n\},$$

if it exists, otherwise $\text{cx}(M) = \infty$. Note that $\text{cx}(M) = 0$ precisely for modules M having finite projective dimension. Moreover, we have $\text{cx}(M) \leq 1$ if and only if the dimensions of the P_n 's are bounded. Clearly, if M is Ω -periodic then $\text{cx}(M) = 1$. The converse is not true in general, see [19] for a counterexample.

The following well-known result will be useful later.

Lemma 4.1. *Let A be a finite-dimensional algebra.*

(a) *Suppose*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is a short exact sequence of A -modules. Then, if two of the modules in the sequence have complexity ≤ 1 , then so does the third.

(b) *Let α be an automorphism of the algebra A , and for any A -module M let M_α denote the A -module with twisted action $m \cdot a := m\alpha(a)$. Then M and M_α have the same complexity.*

To prove Theorem 1.1, we will use the following standard construction which we recall here for the convenience of the reader.

Lemma 4.2. *(Universal extension) Let A be a finite-dimensional algebra, and let X be an A -module with $\text{Ext}_A^1(X, X) = 0$. Moreover, let V be an A -module such that $n := \dim \text{Ext}_A^1(X, V) > 0$. Then there exists a short exact sequence*

$$0 \rightarrow V \rightarrow U \rightarrow X^n \rightarrow 0$$

for which $\text{Ext}_A^1(X, U) = 0$.

Proof. ([6], Lemma 2.1) We choose a basis of $\text{Ext}_A^1(X, V)$, say e_1, \dots, e_n . Then we construct the short exact sequence

$$0 \rightarrow V \rightarrow U \rightarrow X^n \rightarrow 0$$

such that the pullback under the i^{th} canonical injection $X \rightarrow X^n$ is e_i . Upon applying $\text{Hom}_A(X, -)$ we get a long exact sequence

$$\dots \rightarrow \text{Hom}_A(X, X^n) \xrightarrow{\delta} \text{Ext}_A^1(X, V) \rightarrow \text{Ext}_A^1(X, U) \rightarrow \underbrace{\text{Ext}_A^1(X, X^n)}_0 \rightarrow \dots$$

Note that the map δ is surjective by construction. Hence, it follows that $\text{Ext}_A^1(X, U) = 0$, as desired. \square

Proof of Theorem 1.1. Any module which is $\Omega^{n+2}\nu$ -periodic has complexity ≤ 1 . So we assume now that A has an indecomposable module V which is not $\Omega^{n+2}\nu$ -periodic.

Let $V = U_0$. We construct inductively modules U_1, U_2, \dots, U_n and short exact sequences

$$(\zeta_i) \quad 0 \rightarrow U_{i-1} \rightarrow U_i \rightarrow (\Omega^{n-i}X)^{r_i} \rightarrow 0$$

for $1 \leq i \leq n$, such that $\text{Ext}_A^1(\Omega^j X, U_i) = 0$ for $n-i \leq j \leq n-1$.

(a) We first construct U_1 . If $\text{Ext}_A^1(\Omega^{n-1}X, V) = 0$ then take $U_1 = V \oplus \Omega^{n-1}X$ and $r_1 = 1$. Otherwise, construct the universal extension (see 4.2)

$$0 \rightarrow V \rightarrow U_1 \rightarrow (\Omega^{n-1}X)^{r_1} \rightarrow 0.$$

Then $\text{Ext}_A^1(\Omega^{n-1}X, U_1) = 0$.

(b) For the inductive step, suppose U_1, \dots, U_{i-1} have been constructed. If $\text{Ext}_A^1(\Omega^{n-i}X, U_{i-1}) = 0$ then take $U_i = U_{i-1} \oplus \Omega^{n-i}X$ and $r_i = 1$. Otherwise, we construct the universal extension

$$0 \rightarrow U_{i-1} \rightarrow U_i \rightarrow (\Omega^{n-i}X)^{r_i} \rightarrow 0.$$

Then by construction we have $\text{Ext}_A^1(\Omega^{n-i}X, U_i) = 0$. Furthermore, for $n-i < j$ we have by the inductive hypothesis that $\text{Ext}_A^1(\Omega^j X, U_{i-1}) = 0$.

Since X is maximal n -orthogonal we have for all $k = 0, 1, \dots, n-1$ that

$$\text{Ext}_A^1(\Omega^k X, X) = \text{Ext}_A^{k+1}(X, X) = 0.$$

In particular, in our situation we then know that

$$\text{Ext}_A^1(\Omega^j X, \Omega^{n-i}X) = \text{Ext}_A^1(\Omega^{j-(n-i)}X, X) = 0.$$

By considering the long exact sequence to the previous universal extension, one concludes that $\text{Ext}_A^1(\Omega^j X, U_i) = 0$, thus completing the inductive step.

The module U_n satisfies

$$\text{Ext}_A^{j+1}(X, U_n) = \text{Ext}_A^1(\Omega^j X, U_n) = 0$$

for $0 \leq j \leq n-1$. Since X is maximal n -orthogonal it follows that U_n belongs to $\text{add}(X)$. In particular, U_n is $\Omega^{n+2}\nu$ -periodic, by Theorem 3.1, and then has complexity ≤ 1 , by Lemma 4.1. Now we use downward induction. In the extension (ζ_n) , the last two terms have complexity ≤ 1 and hence so does the first term, that is, U_{n-1} , by Lemma 4.1. For the inductive step, suppose U_j has complexity ≤ 1 , then the last two terms in the sequence (ζ_j) have complexity ≤ 1 and hence so does U_{j-1} , again by 4.1. The last step shows that V has complexity ≤ 1 . \square

5. CONCLUDING REMARKS AND OPEN QUESTIONS

5.1. We have proved that if A is selfinjective and A has a maximal n -orthogonal module then all A -modules have complexity ≤ 1 . For algebras of finite representation type, this is no restriction. On the other hand, it is a very strong restriction in general. One would like to know which algebras of infinite representation type have this property, and whether any such algebra has a maximal n -orthogonal module.

Algebras for which every module has $\Omega^3\nu$ -period ≤ 2 were classified in [8]. The list consists of the preprojective algebras of Dynkin type, then one series of algebras denoted by $P(\mathbb{L}_n)$ (where $n \geq 2$) which have precisely one simple module with self-extensions, and otherwise certain deformations of these algebras. By [15], preprojective algebras do have maximal 1-orthogonal modules. On the other hand, they do not have maximal n -orthogonal modules for $n \geq 2$ since $\text{Ext}^2(M, M) \neq 0$ for all non-projective modules. It is also easy to see that $P(\mathbb{L}_2)$ (which is of finite type) does not have a maximal 1-orthogonal module. We do not know whether or not $P(\mathbb{L}_n)$ for $n \geq 3$, or the deformations of preprojective algebras have maximal 1-orthogonal modules.

Tame selfinjective algebras for which all modules are periodic can be found in [4], [5]. Moreover, there are as well the algebras of quaternion type in [9].

5.2. For algebras A of quaternion type, we can show that they can not have a maximal n -orthogonal module, for any $n \neq 2$. Recall that by definition A is symmetric (hence $\tau \cong \Omega^2$ and $\nu \cong \text{id}$) and for any A -module M one has $\Omega^4 M \cong M$ (see [9]). First, for any non-projective indecomposable A -module M we get from Lemma 3.2

$$\text{Ext}_A^3(M, M) \cong \text{Ext}_A^1(M, \Omega^5 M) \cong \text{Ext}_A^1(M, \Omega M) \cong \underline{\text{Hom}}_A(\Omega M, \Omega M) \neq 0.$$

In particular, A cannot have a maximal n -orthogonal module where $n \geq 3$. Secondly, suppose X is a maximal 1-orthogonal A -module. Then by Lemma 3.2 we have

$$\text{Ext}_A^1(X, \Omega^{-1} X) \cong \text{Ext}_A^1(X, X) = 0.$$

Hence, for any non-projective indecomposable summand Y of X , we also have $\Omega^{-1} Y \in \text{add}(X)$. But on the other hand,

$$\text{Ext}_A^1(\Omega^{-1} X, X) \cong \underline{\text{Hom}}_A(X, X) \neq 0,$$

which contradicts the maximal 1-orthogonality of X .

5.3. We do not know any algebra of infinite type for which all modules have complexity ≤ 1 but which has modules which are not $\Omega^{n+2}\nu$ -periodic. Such algebra, if it exists, would have very unusual homological properties. For example if for such algebra, the Nakayama automorphism ν has finite order, then the finite generation properties Fg1 and Fg2 in [10] must fail, since those imply that Ω -periodicity is the same as having complexity one.

5.4. Maximal n -orthogonal modules for an algebra A have all projective indecomposable and all injective indecomposable A -modules as direct summands. Since $\text{Ext}_A^i(X, X) = 0$ for $i = 1, \dots, n$ for any maximal n -orthogonal module, there cannot be any non-split extensions between injective and projective A -modules. Therefore, when searching for maximal n -orthogonal modules it seems very natural to consider selfinjective algebras, as we did in this paper. However, there are non-selfinjective algebras having maximal n -orthogonal modules. The examples we know are all of finite type, and hence its modules also are periodic. It would be interesting to know whether there can exist maximal n -orthogonal modules for non-selfinjective algebras A for which not all A -modules are of complexity at most 1.

REFERENCES

- [1] M. Auslander, Representation dimension of Artin algebras. Queen Mary College, Mathematics Notes, University of London, 1971. Also in: I. Reiten, S. Smalø, Ø. Solberg (Eds.), Selected works of Maurice Auslander, Part I, Amer. Math. Soc., Providence, RI, 1999, 505-574
- [2] M. Auslander, I. Reiten, Representation theory of Artin algebras III. *Comm. Algebra* 3 (1975), 239-294
- [3] M. Auslander, I. Reiten, S. Smalø, Representation theory of Artin algebras. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1995
- [4] J. Białkowski, A. Skowroński, Selfinjective algebras of tubular type. *Colloq. Math.* 94 (2002), no. 2, 175-194
- [5] J. Białkowski, A. Skowroński, On tame weakly symmetric algebras having only periodic modules. *Arch. Math. (Basel)* 81 (2003), no. 2, 142-154
- [6] K. Bongartz, Tilted algebras. *Representations of algebras* (Puebla, 1980), pp. 26-38, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981
- [7] S. Brenner, M. C. R. Butler, A. D. King, Periodic algebras which are almost Koszul. *Algebr. Represent. Theory* 5 (2002), 331-367
- [8] J. Białkowski, K. Erdmann, A. Skowroński, Deformed preprojective algebras of generalized Dynkin type. *Trans. Amer. Math. Soc.*, to appear. Preprint available at <http://www.mat.uni.torun.pl/preprints/data/000085/pp000085.pdf>
- [9] K. Erdmann, Blocks of tame representation type and related algebras. *Lecture Notes in Mathematics*, 1428. Springer-Verlag, Berlin, 1990.
- [10] K. Erdmann, M. Holloway, N. Snashall, Øyvind Solberg, R. Taillefer, Support varieties for selfinjective algebras. *K-Theory* 33 (2004), 67-87
- [11] K. Erdmann, O. Kerner, On the stable module category of a self-injective algebra. *Trans. Amer. Math. Soc.* 352 (2000), no.5, 2389-2405

- [12] K. Erdmann, N. Snashall, Hochschild cohomology of preprojective algebras I. *J. Algebra* 205 (1998), 413-434
- [13] K. Erdmann, N. Snashall, Preprojective algebras of Dynkin type, periodicity and the second Hochschild cohomology. In: I. Reiten, S. Smalø, Ø. Solberg (eds.), *Algebras and Modules II*, CMS Conf. Proc. 24, American Math. Soc., Providence, RI, 1998, pp. 183-193
- [14] S. Fomin, A. Zelevinsky, Cluster algebras I. Foundations. *J. Amer. Math. Soc.* 15 (2002), no. 2, 497-529
- [15] C. Geiss, B. Leclerc, J. Schröer, Rigid modules over preprojective algebras. *Invent. Math.*, to appear. [ArXiv:math.RT/0503324](#)
- [16] K. Igusa, G. Todorov, On the finitistic global dimension conjecture for Artin algebras. *Representations of algebras and related topics*, 201-204, Fields Inst. Commun., 45, Amer. Math. Soc., Providence, RI, 2005
- [17] O. Iyama, Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories. Preprint (2004). [ArXiv:math.RT/0407052](#)
- [18] O. Iyama, Higher dimensional Auslander correspondence. Preprint (2004). [ArXiv:math.RT/0411631](#)
- [19] S. Liu, R. Schulz, The existence of bounded infinite DTr -orbits. *Proc. Amer. Math. Soc.* 122 (1994), 1003-1005

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